Math.131 Final Exam

A

Adama science and Technology University
School of natural sciences
Department of mathematics
Final-Exam of Applied Mathematics I (Math 131)

Time allowed: $2^{1/2}$ hours

Program: Regular

Instruction: - Before you start, please check that this exam consists of two parts and 6 pages (without the cover page). If there is a missing page, inform the invigilator right away, and any complain about the missing pages is completely unacceptable out of the exam room.

Name ______ **ID No** ______________ **Group/add** ______________

Part I (22%)	Part II (23%)					Total (45%)
	Q_1 (4%)	Q_2 (4%)	Q_3 (5%)	Q_4 (5%)	Q_5 (5%)	

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Part IShort Answer Items (22%)

Write the most simplified answer on the space provided. (Each blank space worth 2pts)

1. Evaluatethe following limits

a) $\lim_{t \to 3} \frac{t\sqrt{t+1}-6}{t-3} =$ $\frac{15}{4}$

b) $\lim_{x \to -\infty} \frac{3-2|x|}{3x-4} =$ $\frac{2}{3}$

c) $\lim_{x \to 0^+} (\sin x)^{\frac{2}{\ln x}} =$ e^2

2. Given $h(x) = \log_5 2x$, then find $\frac{d^2h}{dx^2}$ $at\ x = 1$. Ans. = $-\frac{1}{\ln 5}$

3. The value of B so that $g(x) = \begin{cases} \frac{\tan 2x}{x}, & if\ x \neq 0 \\ B, & if\ x = 0 \end{cases}$ is continuous at $x = 0$= $B = 2$

4. The equation of line perpendicular to the curve of $y^3 = x^4y^2\tan x - 4$ at (0,2) is $x = 0$

5. The radius of a spherical balloon is decreasing at the rate of $\frac{1}{10} m/_s$. When the radius is 5m, at wha rate does the

i. The volume changing $V = 10\pi\ m^3/s$ at this rate decreasing

ii. The surface area decreasing $A = 4\pi\ m^2/s$ at this rate decreasing

6. Find all critical number(s) of the function $f(x) = \sqrt{x} - \frac{4}{x-1}$. Ans. = $x = 0$, $x = \sqrt{17} - 4$

7. The number c that satisfies the Mean Value Theorem for the function $f(x) = \frac{-2}{x}$ on [1,3] is $\sqrt{3}$ $\pm\sqrt{6}$?

8. The formula for the n^th^ order derivative for $f(x) = \ln(1-x)$ *is* $\frac{(-1)^{n+1}\, n!}{(x-1)^n}$

$f(x) = \ln(1-x)$

$f'(x) = \frac{1}{1-x}(-1) = \frac{1}{x-1}$

$f''(x) = \frac{-1}{(x-1)^2}$

$f'''(x) = \frac{2}{(x-1)^3}$

$f''''(x) = \frac{-6}{(x-1)^4}$

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$f^n(x) = \frac{(-1)^{n+1}\, n!}{(x-1)^n}$

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Part II: workout Items (23%)

Workout the following problems by showing all the necessary steps clearly and neatly.

1. Using $\in - \delta$ definition show that (4pts)

$$\lim_{x \to 3} 2x^2 - 1 = 17$$

for all $\forall \varepsilon$, there exist some δ $\exists \delta$

such that $0 < |x-3| < \delta$, then $|2x^2-1-17| < \varepsilon$

but $|2x^2-18| < \varepsilon$

$2|x^2-9| < \varepsilon$

$2|(x-3)(x+3)| < \varepsilon$

Let $\delta_1 = 1$ $\quad |x-3| < \delta_1$

$|x-3| < 1$

$-1 < x-3 < 1$ $\quad 0 < x < 4$

$3 < x+3 < 7$

$3 < |x+3| < 7$

$2|(x-3)|x+3|| < \varepsilon$

$2|(x-3)\,7| < \varepsilon$

$14|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{14}$. Since $0 < |x-3| < \delta$

We can choose. $\delta = (1, \frac{\varepsilon}{14})$

2. Using definition compute the derivative ($f'(x)$) of (4pts)

$f(x) = ln(x+3)$

Sol^n

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{\ln(x+h+3) - \ln(x+3)}{h}$$

$$\lim_{h\to 0} \frac{\ln((x+3)+h) - \ln(x+3)}{h}$$

Since, $\ln a - \ln b = \ln \frac{a}{b}$

$$\lim_{h\to 0} \frac{\ln \frac{((x+3)+h)}{x+3}}{h} = \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x+3}\right)}{h}$$

Since $\frac{1}{k} \ln a = \ln a^{\frac{1}{k}}$

$$\lim_{h\to 0} \ln\left(1+\frac{h}{x+3}\right)^{\frac{1}{h}}$$

Consider x as constant

let $x+3 = m$

$$\lim_{h\to 0} \ln\left(1+\frac{h}{m}\right)^{\frac{1}{h}}$$

Let $\frac{h}{m} = \xi$ $\quad h = \xi m \quad h\to 0 \quad \xi \to 0$

$$\lim_{\xi\to 0} \ln(1+\xi)^{\frac{1}{\xi}\cdot\frac{1}{m}}$$

$$\ln\left(\lim_{\xi\to 0}(1+\xi)^{\frac{1}{\xi}}\right)^{\frac{1}{m}}$$

Since $\lim_{\xi\to 0}(1+\xi)^{\frac{1}{\xi}} = e$

$$\ln e^{\frac{1}{m}} = \frac{1}{m}\ln e$$

$\ln e = \log_e e = 1$

$$= \frac{1}{m}$$

Where $m = x+3$

$$= \frac{1}{x+3}$$

Therefore $f'(x)$ of $\ln(x+3) = \frac{1}{x+3}$

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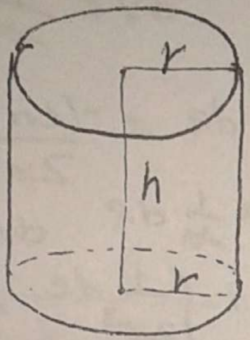
3. A manufacturing company needs to make a packed cylindrical can that will hold of liquid of volume $2000\pi\ cm^3$. Determine the dimensions (radius and height) of the can that will minimize the amount of material needed to manufacture the can. (5pts)

Solution



The Volume $V = \pi r^2 h = 2000\pi\ cm^3$

The surface are $A = \pi r^2 + \pi r^2 + 2\pi r h$
$= 2\pi r^2 + 2\pi r h$

The Area is minimize or minimize amount of material needed

$$h = \frac{2000\pi}{\pi r^2} = \frac{2000}{r^2}$$

$$A = 2\pi r^2 + 2\pi r \cdot \left(\frac{2000}{r^2}\right) = 2\pi r^2 + \frac{4000\pi}{r}$$

$$\frac{dA}{dr} = (A)' = 0 \quad 4\pi r - \frac{4000\pi}{r^2} = 0$$

$$\frac{4\pi r^3}{4\pi} - \frac{4000\pi}{4\pi} = 0$$

$$r^3 - 1000 = 0 \quad r = \sqrt[3]{1000} = \underline{\underline{10\ cm}}$$

The height $h = \frac{2000}{r^2} = \frac{2000}{(10)^2} = \underline{\underline{20}}\ cm$

Therefore dimensions $r = 10\,cm$ ✓
$h = 20\ cm$ ✓

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$\frac{x^{-3+1}}{-3+1} = \frac{1}{-2x^2}$

4) Evaluate the following integral

a) $\int \frac{(\ln x)^2}{x^3}dx$ (2.5pts)

$\int \frac{(\ln x)^2}{x^3}dx = \int \frac{(\ln x)^2}{x^3}dx$

Integration by Part

$V = (\ln x)^2 \quad dV = 2\ln x \cdot \frac{1}{x}dx \quad dV = \frac{1}{x^3}dx, \quad V = \frac{-1}{2x^2}$

$VdV = UV - \int VdU = \frac{-(\ln x)^2}{2x^2} - \int \frac{-1}{2x^2}\frac{2\ln x}{x}dx = \frac{-(\ln x)^2}{2x^2} + \int \frac{\ln x}{x^3}dx$

Integrate again by part $\quad U = \ln x \quad dU = \frac{1}{x}dx \quad dv = \frac{1}{x^3}dx \quad V = \frac{-1}{2x^2}$

$\int VdV = \int \frac{\ln x}{x^3} = -\frac{\ln x}{2x^2} - \int \frac{-1}{2x^2}\cdot\frac{1}{x}dx = -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3}dx = \frac{-\ln x}{2x^2} - \frac{1}{4x^2}$

Therefore $\boxed{\int \frac{(\ln x)^2}{x^3}dx = \frac{-(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2} + c}$ → Answ

b) $\int \frac{x^4+x^3+x^2+1}{x^2+x-2}dx$ (2.5pts)

The 1st step is divide $x^4+x^3+x^2+1$ by x^2+x-2

$$
\begin{array}{r l}
 & x^2+3 \\
x^2+x-2 \,) & x^4+x^3+x^2+1 \\
 & -(x^4+x^3-2x^2) \\
 & +0+0+3x^2+1 \\
 & -(3x^2+3x-6) \\
\end{array}
$$

remainder: $-3x+7$

$\int \frac{x^4+x^3+x^2+1}{x^2+x-2}dx = \int (x^2+3)dx + \int \frac{-3x+7}{x^2+x-2}dx$

$= \frac{x^{2+1}}{2+1} + 3x + \int \frac{-3x+7}{(x-1)(x+2)}dx$ → by Partial fraction

$\int \frac{-3x+7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$Ax + 2A + Bx - B = -3x + 7$

$\boxed{A+B=-3} \quad \boxed{2A-B=7}$

$B = 2A-7$

$A + 2A - 7 = -3$

$3A = -3+7 = 4$

$\boxed{A = \frac{4}{3}}$

$b = 2(\frac{4}{3}) - 7$

$b = \frac{8}{3} - 7 = \frac{8-21}{3} = \boxed{\frac{-13}{3} = b}$

See next page

$$\therefore \int \frac{-3x+7}{x^2+x-2} = \int \left(\frac{4}{3} \cdot \frac{1}{x-1} + \frac{-13}{3} \cdot \frac{1}{x+2} \right) dx$$

$$= \frac{4}{3} \ln|x-1| - \frac{13}{3} \ln|x+2|$$

Therefore

$$\boxed{\int \frac{x^4+x^3+x^2+1}{x^2+x-2} dx = \frac{x^3}{3} + 3x + \frac{4}{3} \ln|x-1| - \frac{13}{3} \ln|x+2| + C}$$

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5) Given $f(x) = \frac{x+1}{x^2}$, then calculate (5pts)

a. The intercepts and asymptotes (if any)
b. The intervals on which f increases and decreases
c. The local extreme values (if any)
d. The concavity and inflection points of f (if any)
e. Sketch the graph of the function f

(a) Intercept and asymptotes

$f(x) = \frac{x+1}{x^2}$

$y = \frac{x+1}{x^2}$

x intersept when $y = 0$ $\quad 0 = \frac{x+1}{x^2}$ $\quad x+1 = 0$ $\quad x = -1$

x intersept $(-1, 0)$

y intersept = has no y-intercept

Vertical Asymptote. $x^2 \neq 0$ $\quad x \neq 0$ or $\lim_{x \to 0} f(x) = \infty$

$x = 0$ is V.A

horizontal Assymptote $\lim_{x\to\infty} f(x) =$ $\lim_{x\to\infty} \frac{x+1}{x^2} = 0$

h.A = $y = 0$

(b) The intervals of on which f increases and decreases.

$f(x) = \frac{x+1}{x^2}$

$f'(x) = \frac{(x+1)'x^2 - (x^2)'(x+1)}{(x^2)^2}$

$f'(x) = \frac{x^2 - 2x^2 - 2x}{x^4}$

$f'(x) = \frac{-x^2 - 2x}{x^4}$

$f'(x) = \frac{-x^2 - 2x}{x^4}$

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$$f'(x) = \frac{-x^2 - 2\cancel{x}}{x^{\cancel{4}}}$$

$$f'(x) = \frac{-x-2}{x^3} = 0$$

find critical point

$$-x-2 = 0$$

$$x = -2$$ is the one critical point

by using sign chart

		-2		0	
$-x-2$	$+ + + + +$	0	$- - - - - -$		$- - -$
x^3	$- - - - -$		$- - - - - -$	0	$+ + + + +$
$\frac{-x-2}{x^3}$	$- - - - -$	0	$+ + + + + +$		$- - - - - -$

f is increasing on $[-2, 0) = -2 \le x < 0$

f is decreasing on $(-\infty, -2] \cup (0, \infty)$

c) The local extreme values (if one)

It has local extreme value at $X = -2$

$$f(x) = \frac{x+1}{x^2}$$

$$f(-2) = \frac{-2+1}{(-2)^2} = \frac{-1}{4}$$

local extreme values $\left(-2, \frac{-1}{4}\right)$

d) The concavity and inflection points of f

$$f'(x) = \frac{-x-2}{x^3}$$

$$f''(x) = \frac{-x^3 - 3x^2(-x-2)}{x^6}$$

$$f''(x) = \frac{-x + 3x + 6}{x^4}$$

$$f''(x) = \frac{2x+6}{x^4}$$

$f''(x) = 0 \to$ inflection points

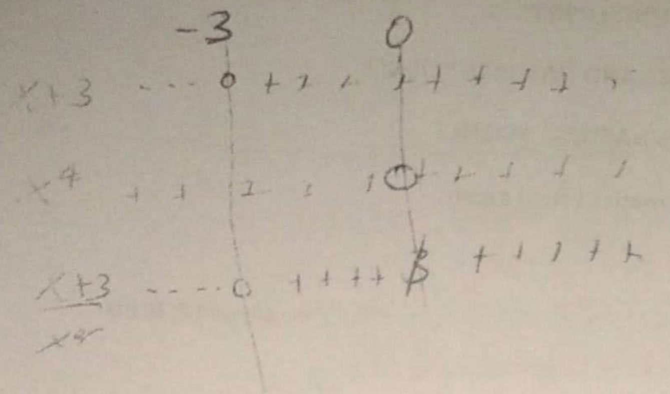
$$\frac{2x+6}{x^4} = 0$$

$$2x+6 = 0$$

$$x = -3$$

$$f(x) = \frac{x+1}{x^2} = f(-3) = \frac{-3+1}{(-3)^2} = \frac{-2}{9}$$

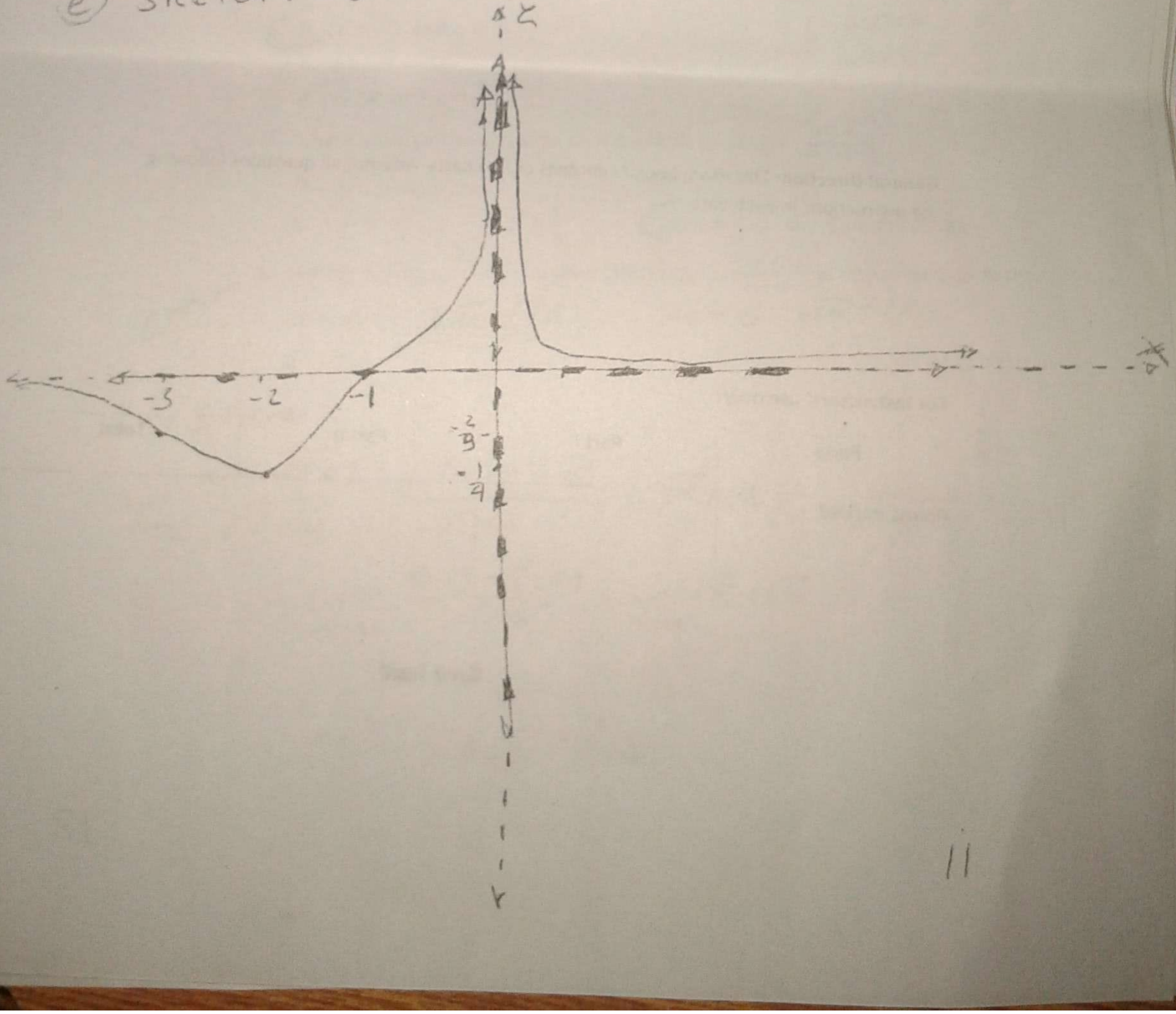
$\left(-3, \frac{-2}{9}\right) \to$ Inflection points he concavity and



f is concave upward $[-3, 0) \cup (0, \infty)$

f is concave downward $(-\infty, -3]$

(e) sketch the graph of the function f



ADAMA UNIVERSITY

SCHOOL OF HUMANITIES AND NATURAL SCIENCE

DEPARTMENT OF NATURAL SCIENCE

Applied Mathematics I final exam

Time allowed 2: 30 hrs

Name:____________________________

ID No:____________________________

Department:________________________

Section:___________________________

General Direction: This exam booklet consists of two parts. Attempt all questions following the instructions in each part.

For instructors' use only!

Parts	Part I	Part II	Total
Points earned			

Good luck!

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PART I: Short answer problems.

Instruction 1: Choose the best answer for each of the following questions.

(2 pts each)

b 1. Which one of the following is true about $f(x) = |1 - x|$

a. f is both continuous and differentiable.

b. f is continuous but not differentiable.

c. f is neither differentiable nor continuous.

d. f is differentiable but not continuous.

b 2. The vertical asymptote to the graph of $f(x) = \frac{1-\cos 2x}{x^2(x+3)(x-2)}$ is/are ____________

a. $x = 0$, $x = -3$ and $x = 2$

b. $x = -3$ and $x = 2$

c. $x = -3$ only

d. $x = 0$ and $x = -3$

b/c 0 is root of $1 - \cos 2x$

$1 - \cos 2x = 0$

$\cos 2x = 1$

$2x = \cos^{-1}(1) = 0$

$x = 0$

C 3. The horizontal asymptote to the graph of $f(x) = \frac{x^2+1}{\sqrt{4x^4+x-1}}$ is ____________

a. $y = 0$ b. $y = \frac{1}{4}$ c. $y = \frac{1}{2}$ d. Doesn't exist

error 4. If $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists then $\lim_{x\to a} \frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}}$ is expressed in terms of $f'(a)$ as $2\sqrt{a}\, f'(a)$

not d. $2\sqrt{a}\, f'(x)$

a. $f'(x)$ b. $2f'(x)$ c. $\sqrt{a}f'(x)$ d. $2\sqrt{a}f'(x)$

Since

$$\lim_{x\to a} \frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} = \frac{f(x)-f(a)}{x-a}(\sqrt{x}+\sqrt{a}) \lim_{x\to a} \frac{f(x)-f(a)}{x-a} (\sqrt{x}+\sqrt{a})$$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} \sqrt{x}+\sqrt{a}$$

$$f'(a) \cdot 2\sqrt{a}$$

$$2\sqrt{a}\, f'(a)$$

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Instruction 2: Give the most simplified answer for each of the following short answer problems. **(2 pts for each blank)**

1. Evaluate the limits.

   a. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$ Ans. $-\frac{1}{4}$

   b. $\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\sin 3x}$ Ans. $\frac{1}{6}$

   c. $\lim_{x\to\infty}\frac{\sqrt{2+4x^2}}{x}$ Ans. 2

   d. If $0 \le f(x) \le c$, *where c is a constant*, then $\lim_{x\to 0} x^2 f(x) =$ ______

2. Find the derivatives of $f(x)$

   a. $f(x) = x\tan x$

   Ans. $\tan x + x\sec^2 x$

   b. $f(x) = 2\sqrt{x} - 3e^x$

   Ans. $\frac{1}{\sqrt{x}} - 3e^x$

   c. $f(x) = \cos(\ln x)$ at $x=1$

   Ans. 0

3. If y satisfies the equation $2xy^2 = 4$ then $\frac{dy}{dx} =$ $\frac{-5}{2x}$

4. If $f(x) = \begin{cases} a^2x - a & for\ x \le 1 \\ 2 & for\ x < 1 \end{cases}$ is continuous then the value(s) of b is (not b)

   It is a

   (are) $-1, 2$

2

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PART II: Work out problems.

Instruction 3: Show all the necessary steps for each of the following work out problems.

1. Using the formal definition of limit show that $\lim_{x \to 1} 6x + 3 = 9$

(3 pts)

for $\forall \varepsilon > 0$. $\exists \delta > 0$

such that

$0 < |x - 1| < \delta$. then $|6x + 3 - 9| < \varepsilon$

$|6x - 6| < \varepsilon$

$6|x - 1| < \varepsilon$

$|x - 1| < \frac{\varepsilon}{6}$

and $0 < |x - 1| < \delta$

we can choose $\delta = \frac{\varepsilon}{6}$

2. Let $g(x) = \begin{cases} x^2 - 1 & for\ x \le 3 \\ ax - b & for\ x > 3 \end{cases}$ then find the values of a and b so that g is differentiable at x=3. **(6 pts)**

Soln

a function is differentiable It is continuous at that Point

$\lim_{x \to 3^+} ax - b = a(3) - b = 3a - b$

$f(3) = 3^2 - 1 = 8$

$f(3) = \lim_{x \to 3^+} f(x)$

$\boxed{3a - b = 8}$ — eqn (1)

and $f'(3^-) = f(3^-) = f'(3)$ b/c It is differentiable at x=

$f'(3^-) = f(3) = (x^2 - 1)'(3) = 2x\big|_3 = 2 \cdot 3 = \underline{\underline{6}}$

$f'(3^+) = (ax - b)' = a \quad f'(3^+) = a$

$\Rightarrow \quad a = 6 \quad$ b/c $f'(3) = 6 = f'(3^+) = a$

from eqn (1)

$3a - b = 8$

$3(6) - b = 8$

$18 - 8 = b$

$\underline{\underline{b = 10}}$

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3. Derive the general formula for the n^{th} order derivative of $f(x) = \frac{1}{x+1}$

(4 pts)

$$f(x) = \frac{1}{x+1}$$

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = -\frac{6}{(x+1)^4}$$

$$\vdots$$

$$f^n(x) = \frac{(-1)^n\, n!}{(x+1)^{n+1}}$$

4. Determine an equation of the line tangent to the graph of the curve $y^4 + x^3y + x = 3$ at $(1,1)$.

(5 pts)

find the slope at $x=1$ $\left.\frac{dy}{dx}\right|_{x=1}$

$$(y^4)' + (x^3y)' + (x)' = (3)'$$

$$4y^3\frac{dy}{dx} + 3x^2y + x^3\frac{dy}{dx} + 1 = 0$$

$$4\frac{dy}{dx} + 3 + \frac{dy}{dx} + 1 = 0$$

$$5\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = -\frac{4}{5}$$

$$y - 1 = -\frac{4}{5}(x-1)$$

$$y = -\frac{4}{5}x + \frac{4}{5} + \frac{5}{5}$$

$$y = -\frac{4}{5}x + \frac{9}{5}$$ ⇒ tangent line to curve

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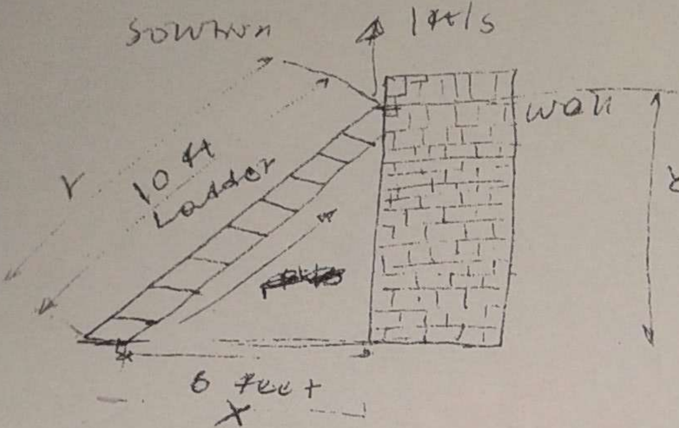
5 Suppose that the top of the 10 feet ladder is being pushed up the wall at the rate of 1 foot per second. How fast is the base of the ladder approaching the wall when it is 6 feet from the wall?
(6 pts)



$$x^2 + z^2 = r^2$$

$$\frac{dx^2}{dt} + \frac{dz^2}{dt} = \frac{dr^2}{dt}$$

$$2x\frac{dx}{dt} + 2z\frac{dz}{dt} = 0$$

since r is constant
r = length of ladder = 10 feet

$$2x\frac{dx}{dt} = -2z\frac{dz}{dt}$$

$$\frac{dx}{dt} = -\frac{z}{x}\frac{dz}{dt}$$

$$\frac{dz}{dt} = 1 \text{ ft/s}$$

$$\frac{dx}{dt} = -\frac{z}{x}\frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{-8}{6}\,1 = -1.33 \text{ ft/s}$$

$z^2 = \sqrt{r^2 - x^2} = \sqrt{100-36} = \sqrt{64} = ~~8ft~~$
$z = 8$ ft

Therefor
The base of ladder approaching the wall at the rate of $\frac{8}{6}$ ft/s = 1.33 ft/s to the wall

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Applied mathematics I (Math.131) final exam. Date 23/02/2009 G.C

Adama University
School of Humanities and Natural Sciences

Time allowed: 3hrs

Part I:-Give short answer for the following problems. (8 pts)

1. Evaluate the following limits (if it exists)

½ . ıı

a) $\lim\limits_{x\to 1}\sqrt{\dfrac{\ln x}{x^4-1}} = \underline{\ \tfrac{1}{2}\ }$ b) $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x-1}\right)^x = \underline{\ e^2\ }$ c) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi}{2}x = \underline{\ \tfrac{2}{\pi}\ }$

2. Suppose $f'(a)$ exists, then $\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{h}$ in terms of $f'(a)$ is $\underline{\ f'(a)\ }$

Part II: - workout problems

Attempt all the problems and show all the necessary steps clearly and neatly.

1. Use $\varepsilon-\delta$ definition to prove that $\lim\limits_{x\to 1}(x^2+x)=2$ (3 pts)

2. Use the definition of derivative to find the derivative of the function (5 pts)

$$f(x) = \ln\left(\frac{1}{\sqrt{x+2}}\right)$$

3. Find the derivative of the function $f(x) = \tan\left(\sqrt{e^{2x}+x^3}\right)$ (3pts)

4. Find the value of a and b so that the function (5pts)

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , \text{if } 0 \le x < \dfrac{\pi}{4} \\ 2x\cot x + b & , \text{if } \dfrac{\pi}{4} \le x < \dfrac{\pi}{2} \\ a\cos 2x - b\sin x & , \text{if } \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

is continuous every where.

5. Let f be a function differentiable at 2, and $f(2)=2$ and $f'(2)=1$.

Find $\lim\limits_{x\to 2}\dfrac{x^2 f(2)-4f(x)}{x-2}$. (5 pts)

6. Find the equation of the normal line to the curve of the equation (4 pts)

$xy + e^{y^2}\ln y = 5\sin x$ at the point (0, 1).

Part II - workout problems Solution

1. for all ε, there exist $\exists$

such that $0 < |x-1| < \delta$, then $|x^2+x-2| < \varepsilon$

but $|x^2+x-2| < \varepsilon$

$|(x+2)(x-1)| < \varepsilon$

let $\delta_1 = 1$

$-1 < x-1 < 1$

$0 < x < 2$

~~$2 < x < 4$~~

$2 < |x+2| < 4$

$|4(x-1)| < \varepsilon$

$|x-1| < \frac{\varepsilon}{4}$

we can choose $\delta = (1, \frac{\varepsilon}{4})$

(2)

$$f(x) = \ln\left(\frac{1}{\sqrt{x+2}}\right)$$

$f'(x)$ by definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{\ln\frac{1}{\sqrt{x+h+2}} - \ln\frac{1}{\sqrt{x+2}}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\ln\left(\frac{1}{\sqrt{x+h+2}} \Big/ \frac{1}{\sqrt{x+2}}\right)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\ln\frac{\sqrt{x+2}}{\sqrt{x+2+h}}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\ln\sqrt{\frac{x+2}{x+2+h}}}{h} = \lim_{h\to 0} \frac{1}{2}\frac{1}{h}\ln\frac{1}{1+\frac{h}{x+2}}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{2}\left(\ln 1^{h} - \ln\left(1+\frac{h}{x+2}\right)^{h}\right)$$

$$= \frac{1}{2}\left(0 - \ln \lim_{h\to 0}\left(1+\frac{h}{x+2}\right)^{h}\right)$$

$$= \frac{1}{2}\left(0 - \ln e^{\frac{1}{x+2}}\right)$$

$$= \frac{1}{2}\left(0 - \frac{1}{x+2}\right) \qquad \ln e = 1$$

$$= \frac{-1}{2x+4}$$

$$f(x) = \tan\left(\sqrt{e^{2x}+x^3}\right)$$

$$f'(x) = \sec^2\left(\sqrt{e^{2x}+x^3}\right) \cdot \frac{1}{2\sqrt{e^{2x}+x^3}} \cdot 2e^{2x}+3x^2$$

$$= \frac{\sec^2\left(\sqrt{e^{2x}+x^3}\right)}{2\sqrt{e^{2x}+x^3}}\left(2e^{2x}+3x^2\right)$$

4) Find the value of a and b

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & \text{if } 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \\ a\cos 2x - b\sin x & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$$

is continuous.

$$\lim_{x \to \frac{\pi}{4}} x + a\sqrt{2}\sin x = f\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\pi}{4}\cot\frac{\pi}{4} + b$$

$$\frac{\pi}{4} + a\sqrt{2}\sin\frac{\pi}{4} = \frac{\pi}{2} + b$$

$$\frac{\pi}{4} + a\sqrt{2} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{2} + b$$

$$\boxed{a - b = \frac{\pi}{4}} \text{ -- eqn } *$$

$$\lim_{x \to \frac{\pi}{2}} (2x\cot x + b) = f\left(\frac{\pi}{2}\right) = a\cos 2\left(\frac{\pi}{2}\right) - b\sin\frac{\pi}{2}$$

$$2 \cdot \frac{\pi}{2}\cot\frac{\pi}{2} + b = a\cos\pi - b\sin\frac{\pi}{2}$$

$$\pi(0) + b = -a - b$$

$$2b = -a \quad \boxed{a = -2b} \quad **$$

$$-2b - b = \frac{\pi}{4} \qquad b = \frac{-\pi}{12} \qquad a = \frac{\pi}{6}$$

Therefore $a = \frac{\pi}{6}$ and $b = \frac{-\pi}{12}$ $f(x)$ is continuous.

5

$f(2) = 2$

$f'(2) = 1$

Find $\lim_{x \to 2} \frac{x^2 f(2) - 4f(x)}{(x-2)}$

The Limit is $\frac{0}{0}$ form

by L hospital rule

$\frac{(x^2 f(2) - 4f(x))'}{(x-2)'}$

$= 2x f(2) \cancel{+ x^2 f(2)} - 4f'(x)$ at $x = 2$

$= 2.2\, f(2) - 4f'(2)$

$= 2.2.2 - 4(1)$

$= 8 - 4$

$= \underline{4}$

6 Find normal line of $xy + e^{x^2} \ln y = 5 \sin x$ at point $(0,1)$

The slope of tangent line

$\frac{d}{dx}(xy + e^{x^2} \ln y = 5 \sin x)$

$y + x\frac{dy}{dx} + 2x e^{x^2} \ln y + \frac{e^{x^2}}{y}\frac{dy}{dx} = 5\cos x$ at $(0,1)$

$1 + 0\frac{dy}{dx} + 2.0 e^{0^2} \ln 1 + \frac{e^{0^2}}{1}\frac{dy}{dx} = 5 \cos 0$

$1 + \frac{dy}{dx} = 5$

$\frac{dy}{dx} = 4$

$\frac{dy}{dx} = 4$ → slope of tangent line

The slope of normal line is $-\frac{1}{4}$

There fore

The equation is $y - 1 = -\frac{1}{4}(x - 0)$ $y = -\frac{1}{4}x + 1$

The equation of normal line is $y = -\frac{1}{4}x + 1$

Applied mathematics I (Math.131) final exam. Date 23/02/2009 G.C

7. If $48m^2$ of sheet metal are to be used in the construction of an open tank with square base, find the dimensions so that the capacity is the greatest possible. (5 pts)

8. Sketch the graph of the function $f(x) = \frac{x^2}{1-x^2}$ by describing all the necessary information. (8 pts)

9. Evaluate a) $\int \sin(\ln x)dx$ b) $\int \frac{1}{x^3+x}dx$ (4 pts each)

10. Suppose f(x) is a function satisfying the following conditions: (6 pts)

a) $f(0) = 2$, $f(1) = 1$

b) f has a minimum value at $x = \frac{5}{2}$, and

c) for all x

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

for some constants a and b.

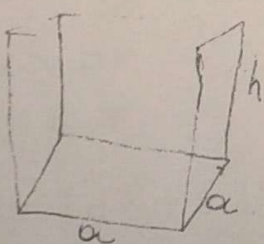
Then determine the constants a and b and the function f(x).

Given

$A_T = 48 m^2$ sheet metal

square base

find dimension if capacity is greatest



Let, Side of base = a

~~height~~ height = h

height

$A_T = a^2 + ah + ah + ah + ah$

$A_T = a^2 + 4ah$ $\qquad a^2 + 4ah = 48$

$\text{Volume} = a^2 h$ $\qquad \dfrac{4ah}{4a} = \dfrac{48 - a^2}{4a} = \dfrac{12}{a} - \dfrac{a}{4}$

$\text{Volume} = a^2\left(\dfrac{12}{a} - \dfrac{a}{4}\right)$ $\qquad h = \dfrac{12}{a} - \dfrac{a}{4}$

$= 12a - \dfrac{a^3}{4}$

$\dfrac{dV}{da} = 12 - \dfrac{3a^2}{4} = 0$

$\dfrac{3a^2}{4} = 12$

$a^2 = \dfrac{12}{3} \cdot 4 = 16$

$a = \sqrt{16} = 4m$

and $h = \dfrac{12}{4} - \dfrac{4}{4} = 3 - 1 = 2m$

The dimension are base $a = 4m$ and $h = 2m$

8) $f(x) = \dfrac{x^2}{1-x^2}$ Sketch the graph of $f(x)$

Vertical Assembtot

V.A $1-x^2=0$ $x=1$ and $x=-1$

horizontal Assembtot the degree of numerator and denominator are equal

H.A $\frac{1}{-1} = -1$

H.A $y = -1$

The x intersept is ((0.0)

The y intercept is (0.0)

Those critical point

$f(x) = \dfrac{x^2}{1-x^2}$

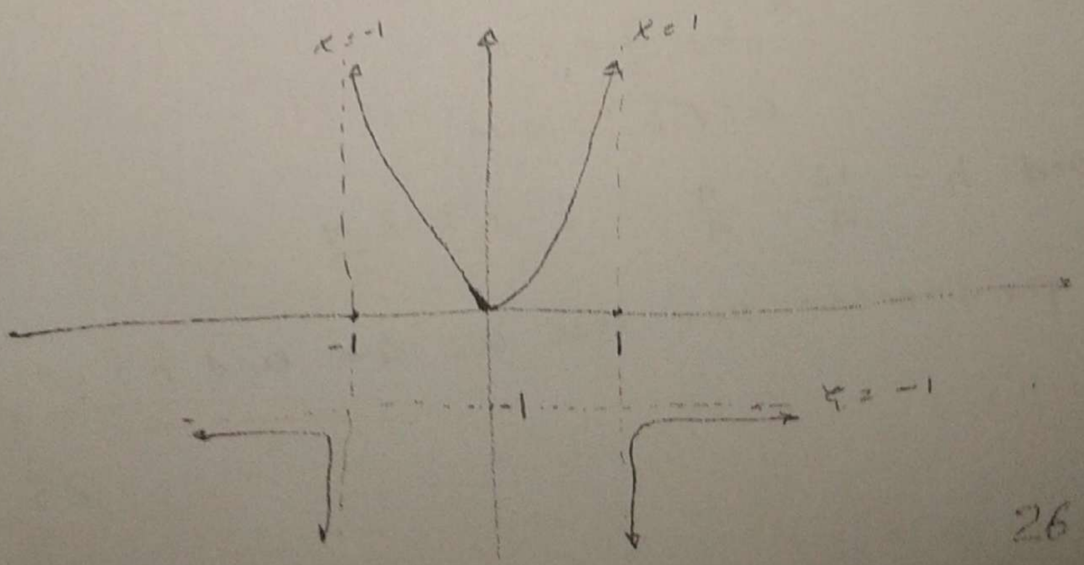
$f'(x) = \dfrac{2x(1-x^2) - (-2x)x^2}{(1-x^2)^2} = \dfrac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \dfrac{2x}{(1-x^2)^2}$

theres no critical point

The inflaction points are

$f''(x) = \dfrac{2(1-2x^2+x^4) - 2(1-x^2)(-2x)2x}{(1-x^2)^2}$

$= \dfrac{2 - 4x^2 + 2x^4 + 8x^2 - 8x^4}{(1-x^2)^2} = \dfrac{2 + 4x^2 - 6x^4}{(1-x^2)^2}$



9)

a $\int \sin(\ln x)\,dx$

let $u = \sin(\ln x)$

$du = \frac{1}{x}\cos(\ln x)$

$dv = dx$

$v = x$

$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$dv = dx$

$v = x$

$u = \cos(\ln x)$

$du = -\frac{\sin(\ln x)}{x}$

$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left(x\cos(\ln x) + \int \sin(\ln x)\,dx\right)$

$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x)$

$\therefore \int \sin(\ln x)\,dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2}$

b $\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x(x^2+1)}\,dx$

by partial ~~d~~ fraction

$\int \frac{1}{x(x^2+1)}\,dx = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$Ax^2+A+Bx^2+Cx = 1$

$A+B=0$

$Cx=0 \quad C=0$

$A=1$

$B=-1$

$\int \frac{1}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx$

$= \ln x - \int \frac{x}{x^2+1}\,dx$

$\int \frac{x}{x^2+1}\,dx \quad$ Let $x^2+1 = v$

$2x\,dx = dv$

$x\,dx = \frac{dv}{2}$

$\int \frac{x\,dx}{x^2+1} = \int \frac{dv}{2} \cdot \frac{1}{v} = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln v = \frac{1}{2}\ln(x^2+1)$

$\int \frac{1}{x^3+x}\,dx = \ln x - \frac{1}{2}\ln x^2+1$

$= \ln x - \ln\sqrt{x^2+1}$

$= \ln \frac{x}{\sqrt{x^2+1}}$

$f(0) = 2$

$f(1) = 1$

$f(x)$ has minimu at $x = \frac{5}{2}$

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

at $x = \frac{5}{2}$ $f'(\frac{5}{2}) = 0$ b/c $x = \frac{5}{2}$, is critical point

$$f'(\tfrac{5}{2}) = \begin{vmatrix} 5a & 5a-1 & 5a+b+1 \\ b & b+1 & -1 \\ 5a+2b & 5a+2b+1 & 5a+b \end{vmatrix} = 0$$

$f'(\frac{5}{2}) = 5a(5ab+b^2+5a+b-5a-2b-1) + (5a-1)(5ab+b^2-5a-2b)$

$+ (5a+b+1)(5ab+2b^2+b-5ab-2b^2-5a-2b)$

$= \cancel{5a}$

$= 25a^2b + 5ab^2 - 5ab - 5a - \cancel{25a^2b} - 5b^2a + 25a^2 + 10ab + 5ab$

$+ b^2 - 5a - 2b + (5a+b+1)(-5a-b)$

$= -10a + 25a^2 + 10ab + b^2 - 2b - 25a^2 - 5ab - 5ab - b^2 - 5a - b$

$f'(\frac{5}{2}) = -15a - 3b = 0$

$-15a = 3b$ $\boxed{b = -5a}$ $\rightarrow$ eqn 1

2nd step find the $|f'(x)|$ and

$f(x) = \int |f'(x)|\, dx + C$

$f(0) = 2$
$f(1) = 1$
$b = -5a$

find $f(x)$ and a and b.

PATR I. True-False Questions: Determine whether the statement is true or false. (1 pt each)

1. ~~True~~ If $\lim_{x\to 1} f(x)=0$ and $\lim_{x\to 1} g(x)=0$, then $\lim_{x\to 1}\left[\frac{f(x)}{g(x)}\right]$ does not exist.

2. True. If $\lim_{x\to a} f(x)=l$, then $\lim_{x\to a^+} f(x)=l$.

3. True. If a function f is differentiable at c, then it is continuous at c.

4. ~~True~~. Let f be a function. If $f(1)<0$ and $f(3)>0$, then there exists a number c between 1 and 3 such that $f(c)=0$.

PATR II. Multiple Choice Questions: Choose the correct answer among the given alternatives and circle your choice. (1.5 pts each)

1. Which of the following is the value of $\lim_{x\to 0}\frac{\sin 3x}{2\sin 5x}$?

   A. $\frac{3}{5}$
   B. $\frac{5}{3}$
   C. $\frac{3}{10}$
   D. $\frac{10}{3}$

2. Which of the following expressions gives the derivative of the function $f(x)=\frac{\sin 3x}{x^2}$?

   A. $\frac{x^2\cos(3x)+2x\sin(3x)}{x^4}$
   B. $\frac{x^2\cos(3x)-2x\sin(3x)}{x^4}$
   C. $\frac{3x^2\cos(3x)-2x\sin(3x)}{x^4}$
   D. $\frac{2x\sin(3x)-3x^2\cos(3x)}{x^4}$

B

$\frac{3x^2\cos 3x-2x\sin 3x}{x^4}$

3. If $y=x^{e^x}$, then $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to:

   A. e
   B. $1/e$
   C. $-e$
   D. $-1/e$

4

30

PART III: Short Answer Questions: Write your answers only on the spaces provided. (Each blank space worth 1.5 pts)

1

1. Evaluate each of the following limits, if the limit exists.

(a) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} =$ $\frac{1}{4}$?

$\frac{1}{2\sqrt{x+1}}$ $\frac{1}{2.2}$

(b) $\lim_{x\to\infty} \frac{x^2+x+1}{(3x+2)^2} =$ $\frac{1}{9}$

$\frac{x^2}{9x^2}$

2. For what values of k is the function

$$f(x)=\begin{cases}3x^3-x^2-kx, & x>1\\ kx-2, & x\le 1\end{cases}$$ continuous at $x=1$?

Answer: $k=2$

3. If $y=x^3e^x$, then $y''=$ $e^x(6x+6x^2+x^3)$

4. Find the equation of the tangent line to the graph of $x^3+y^3=9$ at the point $(1,2)$.

Answer: $y=-\frac{1}{4}x+\frac{9}{4}$, $m=-\frac{1}{4}$

$y=-\frac{1}{4}x+\frac{9}{4}$

$y = x^3e^x = 3x^2e^x + x^3e^x$

$= 6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x$

$(x^3e^x)'' = 6xe^x + 6x^2e^x + x^3e^x$

2

31

33

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PART IV: Work out problems: Show the necessary steps clearly and legibly.

1. Using $\varepsilon-\delta$ definition, prove that

$$\lim_{x\to 1}(3x+2)=5$$ (3 pts)

3

Let $\varepsilon>0$ be given, we went to find $\delta>0$ such that $0<|x-1|<\delta$ then $|f(x)-L|<\varepsilon$.

$\Rightarrow |(3x+2)-5|<\varepsilon$

$\Rightarrow |3x+2-5|<\varepsilon$

$\Rightarrow |3x-3|<\varepsilon$

$\Rightarrow \frac{3|x-1|}{3}<\frac{\varepsilon}{3}$

$\Rightarrow |x-1|<\frac{\varepsilon}{3}$

Choose $\delta=\frac{\varepsilon}{3}$. There for by deffinition.

$\lim_{x\to 1}(3x+2)=5$.

2. Show that the equation $\sqrt{x-8}=\frac{1}{x+5}$ has a solution between 8 and 9. (3 pts)

Ans,

$\sqrt{x-8}=\frac{1}{x+5} \Rightarrow \frac{1}{x+5}-\sqrt{x-8}=0$

$f(x)=\frac{1}{x+5}-\sqrt{x-8}$

If $f(8)$ and $f(9)$ have opposite sign the eqn of $f(x)$ have solution b/n 8 and 9

Then $f(8)=\frac{1}{8+5}-\sqrt{8-8}=\frac{1}{13}-0=\frac{1}{13}$ (+ve) positive

$f(9)=\frac{1}{9+5}-\sqrt{9-8}=\frac{1}{14}-1=-\frac{13}{14}$ (-ve) negative

$f(8)$ and $f(9)$ have opposite sign

There fore $\sqrt{x-8}=\frac{1}{x+5}$ has solution between 8 and 9

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3. A particle moves along a straight line with an equation of motion $S(t) = 3 - 2t + 4t^2$ where S is measured in meters and t in seconds. Find (4 pts)

(a) The average velocity on [1,3].

(b) The instantaneous velocity when $t = 1$.

Given: $S(t) = 3 - 2t + 4t^2$
$t = 1sec.$

Soln

a, $V_{av} = \frac{\Delta v}{\Delta t} = \frac{S(3) - S(1)}{t(3) - t(1)}$,

$= \frac{33m/s - 5m/s}{(3-1)sec}$

$= \frac{28m/s}{2sec}$

$= 14m/s$

But, $S(3) = 3 - 2(3) + 4(3)^2$
$= 3 - 6 + 36$
$= -6 + 39$
$= 33$

$S(1) = 3 - 2(1) + 4(1)^2$
$= 3 - 2 + 4 = -2 + 7$
$= 5$

4

b, $\frac{dv}{dt} = -2 + 8t$ when $t = 1sec$
$= 8t - 2$
$= 8(1) - 2$
$= 6m/s$

4. If the area of a circle is changing at a rate of $6\ m^2/_{sec}$, how fast is the radius changing when $r = 3m$? (4 pts)

Soln

$A = \pi r^2$

$\frac{dA}{dt} = 6\ m^2/sec$

$\frac{dr}{dt} = ?$

$\frac{dA}{dt} = \frac{d(\pi r^2)}{dt}$

$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

$6m^2/sec = \pi\ 2(3m) \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{6m^2/sec}{6\pi m}$

$\frac{dr}{dt} = \frac{1}{\pi}\ m/sec$

Therefore radius changing at a rate of $\frac{1}{\pi}$ m/sec

33